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ON THE SOLUTIONS OF ORDINARY LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

BY GEORGE D. BIRKHOFF

Introduction. Given a linear differential equation of the third order

$$(1)\qquad y''' + py'' + qy' + ry = 0, \qquad (a \leqq x \leqq b),$$

in which the coefficients p, q, r are real functions of x, continuous together with their derivatives of all orders, but otherwise unrestricted; what is the general character of the solutions and how do they depend on these coefficients? This rather interesting question lies almost wholly untreated, and it is the aim of the present paper to consider it.

The analogous question in the case of an equation of the first order

$$y' + py = 0$$

is at once answered. By means of the explicit form of the general solution

$$y = c\, e^{-\int p dx}, \qquad (c, \text{ a constant}),$$

we see first that *y cannot vanish*, unless it does so identically. Also any function y_1 which does not vanish is the solution of such an equation, namely

$$y' - \frac{y_1'}{y_1} y = 0.$$

Thus the general characterization of the solutions has been obtained. The exact nature of the dependence of the solutions upon the coefficient p is exhibited by the explicit formula.

If the equation be of the second order,

$$y'' + py' + qy = 0,$$

the general solution has the form

$$c_1 y_1 + c_2 y_2, \qquad (c_1, c_2, \text{ constants}),$$

where y_1 and y_2 are any pair of linearly independent particular solutions. It is an immediate consequence of the fundamental existence theorem for ordinary differential equations that

$$W = y_1 y_2' - y_2 y_1'$$

cannot vanish. For suppose that this expression did vanish for $x = \xi$. In this event we could determine a and β, not both zero, so that the two equations

$$a y_1(\xi) + \beta y_2(\xi) = 0, \qquad a y_1'(\xi) + \beta y_2'(\xi) = 0$$

would hold. The solution

$$y = a y_1 + \beta y_2$$

and its first derivative would then vanish at $x = \xi$. But, by the existence theorem referred to, there is but one such solution, namely $y = 0$. Accordingly y_1 and y_2 would be linearly dependent, contrary to our hypothesis. Thus W cannot vanish. Furthermore if y_1 and y_2 are any functions for which $W \neq 0$, these functions will form a pair of linearly independent solutions of the linear differential equation of the second order

$$\begin{vmatrix} y'' & y' & y \\ y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \end{vmatrix} = 0,$$

the coefficient of y'' being precisely $-W$. This coefficient can be made unity by division of both members of the equation by $-W$.

The analytic condition $W \neq 0$, which has been shown to characterize y_1 and y_2, tells us that the ratio of y_1 to y_2 increases or decreases throughout the interval considered. Let now y_1 and y_2 be regarded as the cöordinates of a point in the projective line. As x increases through the interval, the point will describe the line in one sense continually. The elementary *Separation Theorem* concerning the roots of $y_1 = 0$ and $y_2 = 0$ is an evident consequence of this fact: *the roots of* $y_1 = 0$ *and* $y_2 = 0$ *will separate each other.* The above results give a characterization of the solutions, and thus answer the first part of our question for a linear differential equation of the second order.

As the most important of the theorems which deal with the dependence of solutions upon the coefficients we cite the following *Comparison Theorem*: suppose two equations

$$y'' + q_1 y = 0 \qquad \text{and} \qquad y'' + q_2 y = 0, \qquad q_1 > q_2,$$

to be given ; then there will be at least one zero of any solution of the first equation between two zeros of any solution of the second, or less precisely, *an increase of* q *in the equation*

$$y'' + qy = 0$$

*brings the zeros of the solutions closer together.**

Very analogous theorems exist in the case of the equation of the third order, and it is with these that we are concerned here.

1. Geometrical Interpretation. If one writes

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix},$$

a necessary and sufficient condition that y_1, y_2, y_3 form three linearly independent solutions of some linear differential equation of the third order (1) is seen to be $W \neq 0$, just as the analogous fact was seen in the case of the second order equation. An explicit formula for W is obtained by noting that one has

$$\frac{dW}{dx} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} = -pW,$$

by (1), so that W is given by the formula

$$W = c\, e^{-\int p dx}, \qquad (c \neq 0). \tag{2}$$

---

*Take for example the equations $y'' + \lambda_1 y = 0$ and $y'' + \lambda_2 y = 0$, where λ_1 and λ_2 are positive constants, λ_1 greater than λ_2. The interval between zeros is fixed in both cases, $\pi/\sqrt{\lambda_1}$ in the first and $\pi/\sqrt{\lambda_2}$ in the second case.

The theorem is more general than at first might appear since any equation of the second order may be given the normal form,

$$y'' + qy = 0,$$

by a simple transformation.

The reader is referred for the proof of this and similar results to articles by Professor Bôcher in the *Bulletin of the American Mathematical Society*, vol. 4, pp. 295-313, pp. 365-376, 1897-1898, as well as to the original paper of Sturm, *Journal de Mathematique*, vol. 1, pp. 106-186, 1836.

To a set of linearly independent solutions y_1, y_2, y_3 of a given equation (1), whose general solution is then

$$c_1y_1 + c_2y_2 + c_3y_3,$$

there will correspond an *integral curve* C defined by the equations

$$w_1 = y_1(x), \qquad w_2 = y_2(x), \qquad w_3 = y_3(x),$$

in which w_1, w_2, w_3 are the homogeneous coördinates of a point in the projective plane referred to a certain triangle of reference. It is clear that any other integral curve D obtained by making a different choice of the three linearly independent solutions will be projectively equivalent to C, since the new solutions can be expressed linearly in terms of y_1, y_2, y_3.

The equation of a tangent line to C is

$$\begin{vmatrix} w_1 & w_2 & w_3 \\ y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \end{vmatrix} = 0.$$

The homogeneous coördinates of this line are therefore

$$(3) \qquad z_1 = \frac{y_2y_3' - y_3y_2'}{W}, \quad z_2 = \frac{y_3y_1' - y_1y_3'}{W}, \quad z_3 = \frac{y_1y_2' - y_2y_1'}{W},$$

the divisor W being so chosen that z_1, z_2, z_3 are solutions of the *adjoint equation* to (1):

$$(4) \qquad y''' - (py)'' + (qy)' - ry = 0.$$

The fact that z_1, z_2, z_3 are solutions of (4) may be verified by substitution. The relation between (1) and (4) is entirely reciprocal in its nature, for equation (1) is also the adjoint to (4). Furthermore, the functions z_1, z_2, z_3 are linearly independent solutions of (4) since otherwise the curve C would reduce to a point or straight line, and y_1, y_2, y_3 would be also linearly dependent. The reader is referred to Darboux, *Théorie des Surfaces,* vol. II, book IV, chap. 5 for a discussion of the properties of the adjoint equation.

The identities

$$
(5) \qquad
\begin{aligned}
y_1 z_1 + y_2 z_2 + y_3 z_3 &= 0,\\
y_1' z_1 + y_2' z_2 + y_3' z_3 &= 0,\\
y_1'' z_1 + y_2'' z_2 + y_3'' z_3 &= 1,
\end{aligned}
$$

are obvious on substitution of the expressions given in (3) for z_1, z_2, z_3. These equations serve also to define z_1, z_2, z_3 in terms of y_1, y_2, y_3.

Returning now to the curve C, we see that it is a *continuous* curve in the projective plane (i.e. if the projective plane be mapped on the ordinary Euclidean plane so that P_0 is a point of C in the finite plane, then C is continuous at P_0 in the ordinary sense) since y_1, y_2, y_3 are continuous functions, not all simultaneously zero ($W \neq 0$). Likewise C has a *continuously turning tangent* in the projective plane (i.e. if the projective plane be mapped on the ordinary Euclidean plane so that P_0 is a point of C in the finite plane, then the direction of the tangent line varies continuously along the curve at P_0), since z_1, z_2, z_3 are also continuous functions, not all simultaneously zero. Thus C is a *regular curve* in the projective plane. The fact that y_1, y_2, y_3 have derivatives in x of all orders may be taken account of by saying that C is *completely regular.*

The significance of the condition $W \neq 0$ is that there are no points of inflection. In fact, by definition, the tangent at such a point has contact with the curve of higher order than the first, and hence if

$$a_1 w_1 + a_2 w_2 + a_3 w_3 = 0$$

is the tangent line, one has simultaneously

$$
\begin{aligned}
a_1 y_1 + a_2 y_2 + a_3 y_3 &= 0,\\
a_1 y_1' + a_2 y_2' + a_3 y_3' &= 0,\\
a_1 y_1'' + a_2 y_2'' + a_3 y_3'' &= 0.
\end{aligned}
$$

This is possible if and only if W vanishes. *The curve C may therefore be any completely regular curve without any point of inflection.* The reader will remember that the only condition on y_1, y_2, y_3 that they form linearly independent solutions of *some* equation (1) is the condition $W \neq 0$.

If we consider the projective plane as lying in the ordinary Euclidean plane, the curve C possesses a tangent which rotates continuously in one direction until the curve passes through the 'line at infinity' which is the image of any arbitrary line in the projective plane. It then returns with a tangent rotating in the reverse direction (consider for example a conic, which has no point of inflection).

A transformation

$$\bar{y} = \lambda y, \qquad (\lambda \neq 0),$$

followed by a division by λ, leaves (1) linear and of course does not change the curve C. By choosing

$$\lambda = e^{-\frac{1}{3}\int p dx},$$

the coefficient of the second term of (1) will reduce to zero and that equation will take on the simple form

$$y''' + qy' + ry = 0, \tag{6}$$

to which the adjoint equation is

$$y''' + qy' + (q' - r)y = 0. \tag{7}$$

It may be observed in passing that this last equation might also have been obtained from (4) by a like transformation in which however

$$\lambda = e^{\frac{1}{3}\int p dx}$$

A simplification introduced by giving our equations this normal form is that W reduces by (2) to a non-vanishing constant.

From this point forth we shall often employ our equations in this simplified form. The curve C may then be looked upon as given either in normalized point coördinates by (6) or in normalized line coördinates by (7).

The above interpretation of the solutions of an ordinary linear homogeneous differential equation by means of a curve is a well-known one. The second and third chapters of Professor Wilczynski's *Projective Differential Geometry* contain a full development of it.

2. General Separation Theorem. The above geometrical representation furnishes us at once with the negative conclusion—not to be seen very readily without use of this representation perhaps—that it is possible to choose (a) the differential equation (1), and (b) three linearly independent solutions y_1, y_2, y_3, of the same, so that the zeros of y_1, y_2, y_3 will succeed each other in any arbitrarily prescribed order. It will suffice to show that there exists a completely regular curve C without points of inflection which cuts the sides of the triangle of reference in any arbitrarily prescribed succession, since such a curve is the integral curve of some equation (1). The existence of such a curve may be readily seen as follows: Let C be drawn up to a point P not on a side of the triangle of reference; the curve C may be extended beyond P without the introduction of points of inflection, so as to cut any assigned side of the triangle. The method of extension is given in the adjoining figure

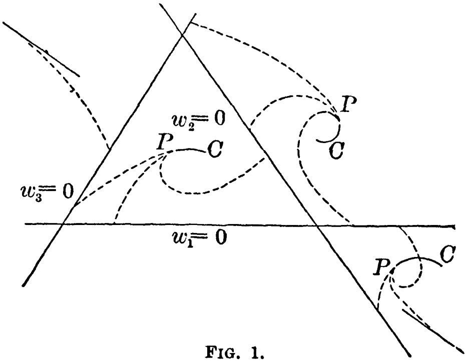

FIG. 1.

(Fig. 1). By a succession of such extensions, C may be made to cut $w_1 = 0$, $w_2 = 0$, $w_3 = 0$ in the desired order.

It is only when we consider the solutions y_1, y_2, y_3 and z_1, z_2, z_3 together that we obtain a Separation Theorem concerning the order of the zeros of these six functions.

GENERAL SEPARATION THEOREM. *Between any two successive zeros of* y_i *(or of* z_i*) there are an odd number of zeros of* y_k *and* z_l *together, where* (i, k, l) *is any permutation of* $(1, 2, 3)$.

Proof. If we can prove that between any pair of successive zeros of y_1 there are an odd number of zeros of y_2 and z_3 together, the theorem will follow by considerations of symmetry and duality. We may suppose without loss of generality that the line $w_1 = 0$ of the projective plane goes over into the line at infinity in the Euclidean plane. The arc of C corresponding to values of x between the two values for which $y_1 = 0$ is then a single *branch* of the curve. The line $w_2 = 0$ is some line in the finite plane. The geometrical interpretation of the condition $y_2 = 0$ is then that the curve C cuts this fixed line $w_2 = 0$; the interpretation of the condition $z_3 = 0$ is that the tangent line is parallel to the line $w_2 = 0$, in other words, passes through the intersection of $w_1 = 0$ and $w_2 = 0$.

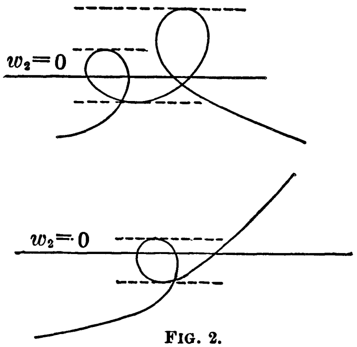

FIG. 2.

Now (Fig. 2) if the arc enters on one side of $w_2 = 0$ in this plane and leaves on the other there are an odd number of zeros of y_2 and an even number of zeros of z_3; otherwise there are an even number of zeros of y_2, and an odd number of zeros of z_3. In either case the theorem holds.

Self-Adjoint Equations. If the integral curve C is self-dual, equations (6) and (7) are identical since the normal form (6) is unique. The condition for this is plainly

$$r = \frac{q'}{2}\cdot$$

In this case, which possesses especial interest, z_1, z_2, z_3 must be linearly expressible in terms of y_1, y_2, y_3:

$$z_1 = c_{11}y_1 + c_{12}y_2 + c_{13}y_3,$$

$$z_2 = c_{21}y_1 + c_{22}y_2 + c_{23}y_3,$$

$$z_3 = c_{31}y_1 + c_{32}y_2 + c_{33}y_3.$$

Since the solutions z_1, z_2, z_3 are linearly independent, the determinant of the coefficients is not zero. Hence one finds from the first equation (5), by substitution of the expressions above for z_1, z_2, z_3, that

$$c_{11}y_1^2 + c_{22}y_2^2 + c_{33}y_3^2 + (c_{23} + c_{32})y_2y_3 + (c_{31} + c_{13})y_3y_1 + (c_{12} + c_{21})y_1y_2 = 0,$$

the equation of a conic unless

$$c_{11} = c_{22} = c_{33} = 0, \qquad c_{23} = -c_{32}, \qquad c_{31} = -c_{13}, \qquad c_{12} = -c_{21}.$$

These conditions cannot all hold since, if they do, the determinant of the coefficients will be zero. *The integral curve C must be an arc of the conic, or* *the entire conic.* In the latter case as x increases from a to b, the corresponding point P may traverse the conic in one sense any number of times, making q complete circuits and a partial circuit.

The above considerations lead us at once to the following theorem:

SEPARATION THEOREM. *Self-Adjoint Case. There exist solutions of any equation* (1) *reducible to self-adjoint form which nowhere vanish in* (a, b). *If* y_1 *and* y_2 *are solutions of the equation with at least one zero in* (a, b), *the zeros of* y_1 *and* y_2 *either separate each other singly or in pairs.*

Proof. In fact, there exist straight lines which do not cut the conic, and therefore solutions which do not vanish, since the condition

$$a_1y_1 + a_2y_2 + a_3y_3 \neq 0$$

is equivalent to the condition that the conic does not cut the line

$$a_1w_1 + a_2w_2 + a_3w_3 = 0.$$

Also, of two straight lines which do cut the conic, say in M_1, M_2 and N_1, N_2, respectively, either M_1 and M_2 separate N_1 and N_2 on the conic, in which case the zeros of the corresponding solutions occur alternately as we traverse the conic; or these pairs do not separate each other, in which case we have first two zeros of the one solution, then two of the other, and so on.

If C is merely an arc of the conic no solution vanishes more than twice in (a, b) although the theorem still holds.

Transformation of the Independent Variable. In obtaining the normal form (6) only a multiplicative transformation of the dependent variable,

$$\bar{y} = \lambda y, \qquad (\lambda \neq 0),$$

was made use of, while it is known that (1) also remains linear when any transformation of the independent variable

$$\bar{x} = \phi(x), \qquad (\phi' \neq 0),$$

is made. From the standpoint of the geometrical interpretation, the second transformation changes the parameter of the curve.

What is the simplest form of the equation obtainable by a suitable combination of both transformations?

First, if the equation be reducible to self-adjoint form the locus of the point (y_1, y_2, y_3) is a conic, and a suitable transformation of the variables will reduce y_1, y_2, y_3 to the form

$$\bar{y}_1 = \cos\bar{x}, \qquad \bar{y}_2 = \sin\bar{x}, \qquad \bar{y}_3 = 1,$$

no matter how many times the integral conic be traversed. Hence a normal form in this case is

$$\bar{y}''' + \bar{y}' = 0, \qquad (\bar{a} \leqq \bar{x} \leqq \bar{b}).$$

Likewise, since the equation (6) may be written

$$y''' + qy' + \left(\frac{q'}{2} + R\right)y = 0,$$

one sees that the transformation which reduces the self-adjoint equation

$$y''' + qy' + \frac{q'}{2}\,y = 0 \tag{8}$$

to the preceding normal form will at the same time reduce (6) to an equation

$$\bar{y}''' + \bar{y}' + \bar{R}\,\bar{y} = 0, \qquad (\bar{a} \leqq \bar{x} \leqq \bar{b}). \tag{9}$$

The Forsyth-Laguerre normal form is

$$\bar{y}''' + \bar{R}\,\bar{y} = 0\,;$$

in this case the self-adjoint equation (8) has been transformed to the equation

$$\bar{y}''' = 0$$

with solutions $1, \bar{x}, \bar{x}^2$. Since the equations

$$w_1 = 1, \qquad w_2 = \bar{x}, \qquad w_3 = \bar{x}^2$$

give us each point of the conic once and only once, this transformation is not proper unless the interval (a, b) is so small that the integral curve of (8) is an arc of the conic.

3. Separation Theorem for Regular Intervals. The somewhat indefinite character of the first Separation Theorem is due in large measure to the fact that the integral curve C may have a very complicated form; in particular it may have double points and double tangents. A *regular interval* (a, b) is defined to be an interval such that the corresponding integral curve C is either an oval, or has no double points or double tangents except where C touches but does not intersect itself. On account of the fact that double points and double tangents are dual notions the interval (a, b) will be regular for the adjoint equation (4) if it is regular for (1) itself, and conversely. It is obvious that any interval can be separated into a finite number of regular intervals.

SEPARATION THEOREM FOR REGULAR INTERVALS. *If* (a, b) *is a regular interval there will exist a solution* y_1 *of* (1) *which is of one sign in* (a, b) *and a family of solutions* $c_2 y_2 + c_3 y_3$ *such that the zeros of any two members of the family separate each other.*

Proof. The theorem is clearly true if the integral curve is an oval, so that this possibility is at once disposed of.

Assume first that the arc AB corresponding to the interval (a, b) does not cut some line of the projective plane, which is taken to be the line at infinity in the Euclidean plane. On account of the hypothesis that (a, b) is regular, it is clear that AB will be *spiraliform* (Fig. 3). We are not excluding the possibility that A or B or both lie on the line at infinity.

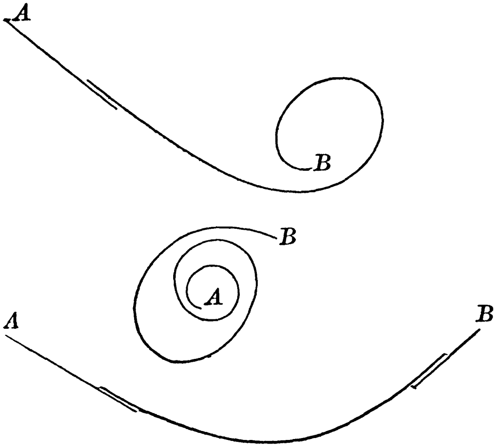

FIG. 3.

We shall now show that there exists such a line in every case. Construct the tangent l at a point P of AB. As P moves from A to B along the arc AB, the line l will not at first cut AP. There must, however, be a least value of x, say ξ, such that for slightly greater values the corresponding tangent l will cut AP; otherwise the tangent line at B does not cut AB and we are brought back to the previous case. Furthermore, the tangent corresponding to $x = \xi$ obviously passes through A. If this particular tangent line be chosen as the line at infinity, it is clear that AP will correspond to a single branch of AB, without inflection point or double point, of course, and that accordingly there exist lines l, which do not touch AP. Now take such a line l, as the line at infinity, and the arc AP will have the form indicated in Figure 4.

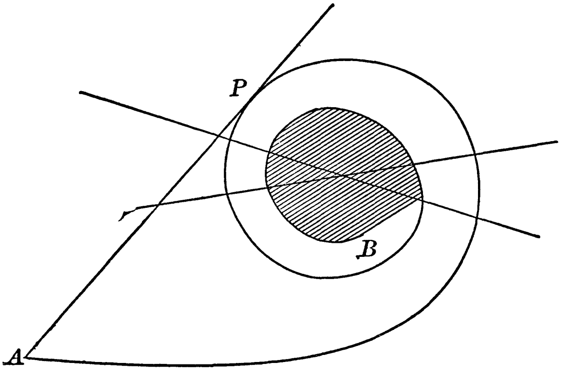

Fig. 4.

From this figure it is clear that, since there are no double points of intersection, the entire arc AB must lie between the arc AP and the tangent at P. Hence this case is reduced to that first considered.

Thus the integral curve C is spiraliform in all cases. It should be noted that an oval is to be looked upon as a limiting form of a spiral.

Any line which does not have a point in common with AB will correspond to a solution y_1, which does not vanish for $a \leqq x \leqq b$. Likewise all the lines through a point in the interior of the integral curve (see shaded region in Fig. 4) will correspond to a linear family

$$c_2y_2 + c_3y_3$$

whose zeros separate each other.

4. An Auxiliary Formula. We shall now derive an auxiliary formula upon which for the most part the further theorems will be based.

Let y_1, y_2, y_3 be three linearly independent solutions of a given linear differential equation (1), and z_1, z_2, z_3 the corresponding solutions of the adjoint equation (4), and let us write

$$\phi(x, \xi) = y_1(x)z_1(\xi) + y_2(x)z_2(\xi) + y_3(x)z_3(\xi). \tag{10}$$

The relations (5) are then

$$\phi(\xi, \xi) = \phi_1(\xi, \xi) = 0, \qquad \phi_2(\xi, \xi) = 1, \tag{11}$$

where our notation is

$$\phi_1(x, \xi) = \frac{\partial \phi}{\partial x}, \qquad \phi_2(x, \xi) = \frac{\partial^2 \phi}{\partial x^2}, \text{ etc.}$$

Take now (1) in the special form (6) and consider the modified equation

$$y''' + qy' + (r + R)y = 0, \tag{12}$$

which may be written in the form of a non-homogeneous linear differential equation

$$y''' + qy' + ry = f = -Ry. \tag{13}$$

The general solution of this equation may be written

$$y = c_1y_1 + c_2y_2 + c_3y_3 + \int_a^x \phi(x, \xi)\, f(\xi)d\xi. \tag{14}$$

In fact by direct differentiation and reduction by means of (11) one finds

$$y' = c_1y_1' + c_2y_2' + c_3y_3' + \int_a^x \phi_1(x, \xi)f(\xi)d\xi,$$

$$y'' = c_1y_1'' + c_2y_2'' + c_3y_3'' + \int_a^x \phi_2(x, \xi)f(\xi)d\xi,$$

$$y''' = c_1y_1''' + c_2y_2''' + c_3y_3''' + f(x) + \int_a^x \phi_3(x, \xi)\, f(\xi)d\xi.$$

Substitute these expressions for y and its derivatives in (13). This equation will then reduce to an identity by virtue of the fact that y_1, y_2, y_3 and $\phi(x, \xi)$ are solutions of (6), which proves that the expression for y given by (14) is the general solution of (13).

The function

$$\eta = c_1 y_1 + c_2 y_2 + c_3 y_3$$

is a solution of (6), and from the above equations we see that it satisfies the conditions

$$(15) \qquad y(a) = \eta(a), \qquad y'(a) = \eta'(a), \qquad y''(a) = \eta''(a).$$

Thus we may rewrite (14) in the final form

$$(16) \qquad y = \eta - \int_a^x \phi(x, \xi)\, R(\xi)\, y(\xi)\, d\xi,$$

in which y and η are corresponding solutions of (12) and (6) respectively, related by equations (15). This is the desired formula, and it represents, of course, nothing more than the solution of a special non-homogeneous linear differential equation.

5. Test for Regular Intervals. A regular interval (a, b) is said to be of the *first kind* if the tangent at A does not meet the arc AB at any second point (see Fig. 4), and of the *second kind* if the tangent at B does not meet the arc AB at any second point. According to this definition a regular interval may be of both kinds. We shall take the interval (a, b) to be of the first kind, stating, however, the results for both cases.

It follows at once from the definition, as has been noted, that if (a, b) is a regular interval for equation (1), it is also a regular interval for equation (4), the three solutions z_1, z_2, z_3 of (4), given by (3), now being interpreted as the coördinates of a point. Let the corresponding point locus be the curve D. Then D is obtained from C by a polar reciprocation and a suitable projection. Furthermore, if (a, b) is regular and of the *first kind* for (1), no line through B is tangent to the curve C at any other point than B (see Fig. 4). Therefore, in the curve D, no point on the tangent through B lies on the curve D, i. e. D is of the *second kind*.

A necessary condition that (a, b) is a regular interval is that *all the points of intersection of the tangent l at any point P of AB lie on one and the same side of P.* Thus in the preceding figure (Fig. 4) all the points of intersection of l lie in AP and none in PB. To prove the condition sufficient we have to show that, if it is satisfied, the integral curve AB can have no double points or double tangents except where the curve C touches but does not intersect itself. Suppose there were such a double point $M = N$ where N lies in MB. Then the tangent at M cuts the curve at N, and the tangent at N cuts the curve at M. This is impossible if the condition is satisfied. If there were such a double tangent at M and N, where N lies in MB, the tangent line in the vicinity of M would cut the integral curve in the vicinity of N, and likewise the tangent line in the vicinity of N would cut the integral curve in the vicinity of M. This is also impossible if the condition is satisfied. Thus the validity of the condition has been demonstrated. Furthermore it is clear from the figure that (a, b) will be regular and *of the first kind* if no point of intersection of the tangent l lies in PB.

The analytic phrasing of the geometric condition can easily be given in terms of the function $\phi(x, \xi)$ defined in (10). For from the equations (11) it follows that the solution of (1) which corresponds to the tangent having a point of contact whose x is ξ, is $\phi(x, \xi)$. Accordingly $\phi(x, \xi)$ will not change sign for $x > \xi$, since the tangent at ξ does not cut the integral curve for points whose x is greater than ξ. Moreover, $\phi(x, \xi)$, regarded as a function of x, has a minimum at the point $x = \xi$, by (11). Therefore the analytic form of the condition is the following: *the interval (a, b) is regular and of the first (second) kind provided that $\phi(x, \xi)$ is positive or zero for $x > \xi\,(x < \xi)$.*

We are now in a position to develop the following test: *If (a, b) is a regular interval of the first (second) kind for the equation* (6) *it is also a regular interval of the same kind for the modified equation* (12), *provided that the inequality $R \leqq 0$ ($R \geqq 0$) obtains.*

Proof. The proof can immediately be made by means of the auxiliary formula (16) in which we take

$$\eta = \phi(x, a),$$

where a remains to be specified. This choice of η is possible since $\phi(x, a)$ is a solution of (6). We obtain then

$$y = \phi(x, a) - \int_a^x \phi(x, \xi)\, R(\xi)\, y(\xi) d\xi, \tag{17}$$

when y is that solution of the modified equation (12) which by (15) satisfies the conditions

$$y(a) = \phi(a, a) = 0, \quad y'(a) = \phi_1(a, a) = 0, \quad y''(a) = \phi_2(a, a) = 1, \tag{18}$$

Accordingly y is precisely the solution of the modified equation (12) which plays the same rôle for (12) that $\phi(x, a)$ does for the equation (6). Therefore if (a, b) is not regular and of the first kind for the modified equation there exists some a such that the corresponding y changes sign for $x > a$. Let x_0 be the least value of x for which this is true. Then we have the equation

$$0 = \phi(x_0, a) - \int_a^{x_0} \phi(x_0, \xi) R(\xi) y(\xi) d\xi.$$

But by hypothesis (a, b) is regular and of the first kind so that $\phi(x_0, a)$ and $\phi(x_0, \xi) (x_0 > \xi)$ are positive or zero. Moreover $R(\xi)$ is negative by hypothesis, and $y(\xi)$ is positive for ξ between a and x_0 by (18). Thus we have the sum of two positive or zero quantities equal to zero. Of these two quantities the second given by the definite integral must be negative; for the first factor $\phi(x_0, \xi)$ of the integrand can only be zero for isolated values of x since $\phi(x_0, \xi)$ is a not identically zero solution in ξ of the adjoint equation to (6), given by

$$y_1(x_0) z_1(\xi) + y_2(x_0) z_2(\xi) + y_3(x_0) z_3(\xi);$$

likewise the third factor of the integral cannot vanish except at x_0 and ξ on account of the way in which x_0 was chosen; finally the remaining factor $R(\xi)$ may be taken to be not identically zero. Hence the above equation is impossible.

The following special case of the test is important: *The interval* (a, b) *is a regular interval of the first (second) kind for an equation*

$$y''' + qy' + \left(\frac{q'}{2} + R\right) y = 0 \tag{19}$$

provided that $R \leqq 0 (R \geqq 0)$ *in the interval* (a, b).

In this case it is only necessary to identify the equation (6) with the self-adjoint equation:

$$y''' + qy' + \frac{q'}{2} y = 0,$$

for which any interval (a, b) is regular and of both kinds, and to state the test for this special case.

Since any equation (6) can be written in the form (19) we have a method of separating any interval into a sequence of regular intervals the first of one kind, the next of the other, and so on, the end points of these intervals being given by those roots of $R = 0$ at which R changes sign.

The regular intervals thus chosen have a special geometric property which however I shall not stop to prove: *The successive osculating conics to the integral curve lie one within the other at all points of such an interval, and at the end points the curve C and the osculating conic hyperosculate.*

6. Intervals of Oscillation. Comparison Theorem. It remains to develop the nature of the dependence of solution and coefficients. Analogous to the notion of the *interval of oscillation* as the distance between successive zeros of a solution of an equation of the second order, is the notion of *forward* and *backward intervals of oscillation.*

Consider the totality of those solutions of (1) which vanish at $x = a$; these solutions will form a two parameter linear family. Suppose that they all vanish again in the interval (a, b). The *least* interval (a, β), where β is greater than a, which has the property that all these solutions vanishing at $x = a$ will vanish again in (a, β) is called the *forward interval of oscillation* at $x = a$. Likewise the *least* interval (β, a) where β is less than a, which has the property that all these solutions vanish again in (β, a) is called the *backward interval of oscillation* at $x = a$.

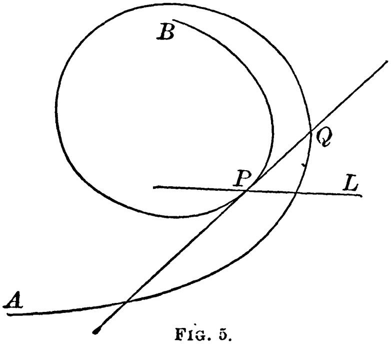

FIG. 5.

It may happen that not all of the solutions which, vanish for $x = a$ vanish again for $x > a$; in this case a forward interval of oscillation at $x = a$ does not exist. Likewise a backward interval of oscillation may not exist.

Let us now consider a regular interval (a, b) which, as before, is taken to be of the first kind. The integral curve AB will then be a spiraliform curve with the point B in its interior (see Fig. 5). For such an interval (a, b) a forward interval of oscillation never exists, unless AB is an oval or touches itself. Thus in the figure let P be any point whose x is a; then the line PL corresponds to a solution which does not vanish for $x > a$, since PL does not meet PB at any second point.

The backward interval of oscillation may however exist. Thus in the figure, the tangent to the curve AB at P first meets the arc AP at Q; if the point Q corresponds to $x = \beta$, then (β, a) is the backward interval of oscillation at a. Any straight line through P meets QP again and this is not true of any shorter arc PQ'. Since $\phi(x,a)$ corresponds to the tangent at a, the quantity β is the greatest root of $\phi(x,a) = 0$ less than a. If the tangent at P does not meet AP again, the backward interval of oscillation does not exist.

It is apparent from this figure that any solution vanishes at most in two distinct points in a backward interval of oscillation (β, a) on such an interval (a, b). Furthermore, if a given solution y vanishes for any value of $x > a$, it will necessarily vanish in (β, a). This is obvious from the figure also.

Similar facts are true of forward intervals of oscillation if (a, b) is of the second kind.

Comparison Theorem. *If* (a, b) *be a regular interval of the first (second) kind for the equation* (6), *and if* $R \leqq 0$ $(R \geqq 0)$, *the backward (forward) interval of oscillation* $(\beta_1,a)[(a,\beta_1)]$ *at* $x = a$ *for* (12) *is smaller than the like interval* $(\beta,a)[(a,\beta)]$ *for* (6).

Proof. To prove this theorem we shall again employ the auxiliary formula (16). As in a preceding proof we shall take

$$\eta = \phi\ (x, a)$$

and thus obtain again (17). Suppose now if possible that the theorem is not true, so that we have $\beta \geqq \beta_1$. As we have already noted, the function analogous to $\phi(x, a)$ for the modified equation (12) is precisely y of (17), which must then vanish for $x = \beta_1$ but not for $\beta_1 < x < a$, by the analytic condition obtained above, since (a, b) is regular for (12) also. The two functions y and $\phi(x, a)$ will therefore be positive throughout (β, a) since neither vanishes within (β, a) and since both are positive in the vicinity of a by (11). Put now $x = \beta$ in equation (17). It reduces to

$$y(\beta) = \int_\beta^a \phi(\beta, \xi)\ R(\xi)\ y(\xi) d\xi,$$

where $\phi(\beta, \xi)$ is positive. But $R(\xi)$ is negative or zero. Therefore in this relation are equated quantities, one positive or zero, the other actually negative. This is impossible and the theorem must hold.

We note the following result contained in the above theorem: *The backward (forward) interval of oscillation for the equation*

$$y''' + qy' + \left(\frac{q'}{2} + R\right) y = 0,$$

where $R \leqq 0$ $(R \geqq 0)$ *is smaller than that for the self-adjoint equation obtained by taking* $R = 0$.

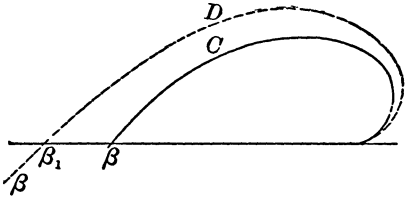

FIG. 6.

Any equation (1) may be reduced to the normal form above by a known transformation. In the present section we have developed the dependence of the solutions on R, by showing that if R is negative a decrease in R decreases the backward interval of oscillation. In the figure (Fig. 6) we have represented by a full line the integral curve C of this equation and by a broken line the integral curve D of the modified equation. If (β, a) represents the backward interval of oscillation at a for the curve C and (β_1, a) the similar interval for D, we have $\beta_1 > \beta$. For the sake of the comparison the curves have been taken with the same initial point and direction. It would now be desirable to give a comparison theorem which stated, in simple terms, the dependence of the interval of oscillation on the function q. If we increase q, however, it may be shown that the interval of oscillation may be either increased or diminished in length. In the self-adjoint case the dependence is more simple:

COMPARISON THEOREM. *Self-adjoint case. Given two equations reducible to self-adjoint form, in particular to*

$$\begin{aligned} &y''' + q_1 y' + \frac{q_1'}{2}\, y = 0, \\ &y''' + q_2 y' + \frac{q_2'}{2}\, y = 0, \end{aligned} \tag{20}$$

such that $q_1 > q_2$ in (a, b); then the backward (forward) interval of oscillation of the first of these equations is less than that of the second.

Proof. In order to prove this theorem we shall prove first that the backward interval (β, a) of oscillation of any self-adjoint equation

$$y''' + qy' + \frac{q'(x)}{2} y = 0 \tag{21}$$

is precisely the interval of oscillation (β, a) for the linear differential equation of the second order

$$z'' + \frac{q}{4} z = 0. \tag{22}$$

Let y be the solution of the equation (21) such that

$$y(a) = y'(a) = 0, \qquad y''(a) = 1. \tag{23}$$

That is, y is chosen so that it is equal to $\phi(x, a)$ by (11). As has been shown, the backward interval (β, a) of oscillation at $x = a$ is given by the largest root of $y = 0$ which is less than a.

Now we have clearly, by triple integration by parts,

$$\int yy'''dx = yy'' - y'^2 + y''y - \int y'''y dx,$$

and by a single integration by parts,

$$\int y(qy')dx = qy^2 - \int (qy' + q'y)ydx.$$

Therefore we have

$$\int y(y''' + qy' + \frac{q'}{2} y)dx = 2yy'' - y'^2 + qy^2 - \int (y''' + qy' + \frac{q'}{2} y)y\, dx.$$

Remembering now that y is a solution of equation (21) which furthermore satisfies conditions (23) we obtain

$$2yy'' - y'^2 + qy^2 = 0.$$

Make the substitution

$$y = z^2, \quad y' = 2\, zz', \quad y'' = 2(zz'' + z'^2),$$

in this last equation, and, on the removal of a factor $4z^3$ it will reduce to the linear differential equation (22). It is clear that, as thus determined, z will be a solution which vanishes at $x = a$. And β will be determined of course as the greatest value of x less than a for which $z = 0$. *Hence the backward (forward) interval of oscillation at a of the given self-adjoint equation* (21) *is given as an interval between two successive zeros, of a solution of the associated linear differential equation* (22) *of the second order.*

The theorem we wish to prove is now an obvious consequence of the Comparison Theorem stated in the introduction.

The above reduction is not new.

7. Oscillatory and Non-Oscillatory Solutions. The preceding paragraphs afford practical methods for obtaining an idea as to the distribution of the zeros of a solution of (1) provided the solution is known to vanish. The following theorem often enables one to determine whether a given solution is oscillatory or non-oscillatory in the interval (a, b) i. e. whether the solution vanishes or not.

THEOREM. *Let* (a, b) *be a regular interval of the first (second) kind for the equation* (6), *and suppose that* $R \leqq 0$ $(R \geqq 0)$. *Let* η *and* y *be solutions of* (6) *and* (12) *respectively, so related that*

$$y(a) = \eta(a), \qquad y'(a) = \eta'(a), \qquad y''(a) = \eta''(a).$$

The inequality

$$|y| < |\eta| \tag{24}$$

obtains for values of x *less (greater) than* a *and within the backward (forward) interval of oscillation at* $x = a$ *until a value of* x *is reached for which* $y = 0$; *the inequality* $|y| > |\eta|$ *obtains for all values of* x *greater (less) than* a *until a value of* x *is reached for which* $\eta = 0$.

Proof. If the inequality (24) fails to hold for $x < a$, it will certainly fail to hold before an x is reached for which one has $\eta = 0$. In fact, otherwise, for this x we have $y = 0$, by (24), and the theorem does not say anything about y for lesser values of x. Therefore if the first part of the theorem does not hold, the inequality

$$|y| \geqq |\eta| \tag{25}$$

is true for some $x < a$, say x_0, in the backward interval of oscillation (β, a), such that neither y nor η vanish in (x_0, a).

But y and η have precisely the properties that they had in the auxiliary formula (16) so that

$$y(x_0) - \eta(x_0) = \int_{x_0} \phi(x_0, \xi)\ R(\xi)\ y(\xi)\ d\xi.$$

The left hand member in this equality is of the same sign as $y(x_0)$ or zero, since (25) holds at $x = x_0$. The right hand member is of opposite sign to $y(x_0)$, since $R(\xi)$ is negative and $\phi(x_0, \xi)$ is positive, x_0 and ξ being contained within the interval of oscillation (β, a) (see Fig. 5). This equality cannot hold and the first part of the theorem is true.

The second part of the theorem is proved in an analogous way. An equation

$$y(x_0) - \eta(x_0) = -\int_a^{x_0} \phi(x_0, \xi)\ R(\xi)\ y(\xi)\ d\ \xi$$

will be obtained if the theorem is not true, in which $x_0 > a$ is an x for which $|y| \leqq |\eta|$ while neither y nor η change sign in (a, x_0). In this case too $R(\xi)$ is negative and $\phi(x_0, \xi)$ is positive. As before, the two members of this equality would have opposite signs.

The relation of the two solutions is given in the figure (Fig. 7).

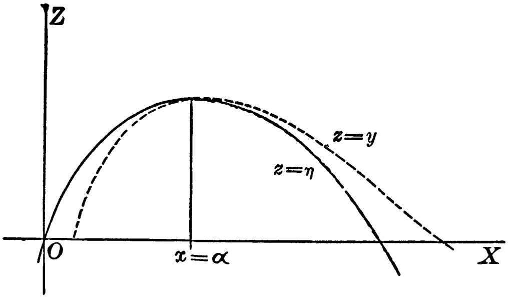

FIG. 7.

The following application is at once obvious: *If* η *and* y *are solutions of the equations* $y''' + qy' + \frac{q'}{2}\,y = 0$ *and of* $y''' + qy' + \left(\frac{q'}{2} + R\right)y = 0$ *respectively, where* $R \leqq 0$ $(R \geqq 0)$, *and are so related that*

$$y(a) = \eta\ (a),\ \ y'(a) = \eta'(a),\ \ y''(a) = \eta''(a),$$

then $|y| < |\eta|$ *for* $x < a (x > a)$ *and* $|y| > |\eta|$ *for* $x > a (x < a)$ *until a value of* x *is reached for which* $y = 0$ *and* $\eta = 0$ *respectively.*

In fact the interval (a, b) under consideration is regular and of either kind for the self-adjoint equation.

8. An Example. It is possible in many cases to form a good idea of the character of the solutions by means of the preceding results. As an illustration we consider the equation

$$y''' + y' + xy = 0,$$

to which the adjoint equation is

$$z''' + z' - xz = 0.$$

We choose y_1, y_2, y_3 as the principal solutions at $x = 0$:

$$\begin{array}{lll} y_1(0) = 0, & y_1'(0) = 0, & y_1''(0) = 1, \\ y_2(0) = 0, & y_2'(0) = 1, & y_2''(0) = 0, \\ y_3(0) = 1, & y_3'(0) = 0, & y_3''(0) = 0. \end{array}$$

The adjoint solutions (see equations (3) then satisfy the conditions

$$\begin{array}{lll} z_1(0) = 1, & z_1'(0) = 0, & z_1''(0) = -1, \\ z_2(0) = 0, & z_2'(0) = -1, & z_2''(0) = 0, \\ z_3(0) = 0, & z_3'(0) = 0, & z_3''(0) = 1. \end{array}$$

The interval $x < 0$ is regular and of the first kind (test for regular intervals, special case) and the interval $x > 0$ is regular and of the second kind. Each of the corresponding parts of the integral curve is therefore spiraliform with the point A corresponding to $x = 0$ in its interior.

Furthermore it is clear that y_1, y_3, z_1, z_3, are even functions of x and that y_2, z_2 are odd functions, since the equation is unchanged if $-x$ be substituted for x. The integral curve must be symmetrical with respect to $w_2 = 0$. Hence the integral curve has the form indicated in the schematic figure (Fig. 8).

The zeros of y_1 and y_2 and indeed of any two members of the linear family $c_1 y_1 + c_2 y_2$, will alternate for $x > 0$ and for $x < 0$ (Separation Theorem for Regular Intervals).

In order to consider the question of the distribution of the zeros of y_1,

y_2 for $x > 0$ we note that the forward interval of oscillation will decrease with increase of x and always be less than that of

$$y''' + y' = 0$$

which is precisely 2π (Comparison Theorem). Hence as x increases the successive zeros of y_1 and of y_2 come at intervals none exceeding 2π and tending to zero.

The function y_3 also vanishes for $x = x_1 > 0$, as a direct computation shows. The successive zeros of $y_3(x)$ will fall of course at intervals less than 2π which tend to zero as x increases.

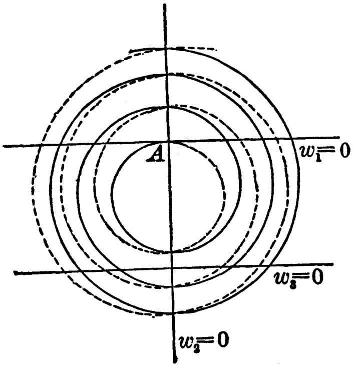

FIG. 8.

9. The Theorems of Liouville. The only known theorems of the kind we have been considering concerning the solution of the third (and higher) order equations appear to be those due to Liouville.*

The type of the equation considered is

$$\frac{d}{dx} K \frac{d}{dx} L \frac{d}{dx} y + \lambda y = 0,$$

where K, L are positive in (a, b) and λ is a parameter. Now when $\lambda = 0$ one finds by direct integration that

$$\phi(x, a) = \int_a^x \frac{c}{L} \int_a^x \frac{1}{K} dx.$$

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* *Liouville's Journal*, vol. 3, pp. 561–614 (1838).

This function does not vanish for $x \neq a$. Hence for $\lambda = 0$ the interval (a, b) is of the first and second kind, the corresponding arc AB has the shape of a part of a convex oval. Suppose now that the equation is reduced to the form (6) and that λ is made to increase. By the test for regular intervals and the Comparison Theorem, the interval (a, b) remains regular of the second kind and the forward intervals of oscillation decrease. In addition, we know by the theorem of §7 that if we fix y, y', y'' at $x = a$, the first greater value of x for which $y = 0$ will decrease with increase of λ.

The results of Liouville are still more definite, such as the following: If we consider a particular solution $y(x)$ for which at $x = a$

$$y = a, \qquad \frac{d}{dx}y = b, \qquad \frac{d}{dx}L\frac{d}{dx}y = c, \qquad a > 0, \qquad b > 0, \qquad c > 0,$$

all of the greater values of x for which y vanishes must decrease with increase of λ.

This result is equivalent to his oscillation theorem which states that there exists one and only one value of λ, say λ_0, for which y vanishes at b and not within (a, b), and one and but one value λ_1, such that y vanishes at b and once within (a, b) and so on.

Liouville's results belong to another category than those which we have obtained. We have given only theorems concerned with any arbitrary increase of the coefficients. A special kind of variation was considered by Liouville which is however of great importance.

PRINCETON UNIVERSITY,
JUNE, 1910.